On primes of the form $n_1^{b_1} + n_2^{b_2} + k$, on average

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Introduction

It is a celebrated theorem of J. B. Friedlander and H. Iwaniec that the polynomial $x^2 + y^4$ assumes infinitely many prime values [FI2]. The literature on the representation of primes by polynomials is vast and we cannot do justice to the history of this important problem beyond directing the reader to [B], [B2], [B3], [BH], [BZ], [BZ2], [BZ3], [DL], [FI], [FI3], [G2], [GT], [H], [HL], [HM], [HM2], [I], [I2], [M2], [M3], [P1], [P2], [P3], [S], [SS] and [W]. The aim of this paper is to follow the spirit of [BZ] in applying the beautiful Hardy-Littlewood circle method to study the problem of representing primes by polynomials of the form $n_1^{b_1} + n_2^{b_2} + k$, on average. Without loss of generality, we make it a convention that $b_2 \geq b_1$. The following conjecture is based on heuristic reasoning similar to that of [BH] and many other papers:

Conjecture.

$$\sum_{n_1, n_2 \leq x} \Lambda(n_1^{b_1} + n_2^{b_2} + k) \sim \mathfrak{S}(b_1, b_2, k) x^2 \tag{1}$$

where

$$\mathfrak{S}(b_1, b_2, k) = \prod_p \frac{p^2 - n_{p,b_1,b_2,k}}{p(p-1)} = \sum_q \frac{\mu(q) \prod_{p|q} (n_{p,b_1,b_2,k} - p)}{q\varphi(q)}, \tag{2}$$

with $n_{p,b_1,b_2,k}$ being the number of solutions to the equation

$$n_1^{b_1} + n_2^{b_2} + k \equiv 0 \bmod p, (n_1, n_2) \in (\mathbb{Z}/p\mathbb{Z})^2.$$

We see that the Hasse-Weil bound [W2] facilitates the convergence of the singular series. Namely, when the curve $n_1^{b_1} + n_2^{b_2} + k = 0$ has genus g, one has, by the Hasse-Weil bound, that

$$\frac{n_{p,b_1,b_2,k}}{p} - 1 = O\left(\frac{g}{p^{1/2}}\right). \tag{3}$$

Therefore, letting $\omega(q)$ denote the number of distinct prime divisors of q and $\tau(q)$ denote the number of divisors of q, and recalling that $\tau(q) \ll q^\epsilon$, we have that

$$\sum_q \frac{\mu(q) \prod_{p|q} (n_{p,b_1,b_2,k} - p)}{q\varphi(q)} \ll \sum_q \frac{g^{\omega(q)}}{\varphi(q) q^{1/2}} \ll \infty. \tag{4}$$

We now state the theorem.

Theorem. *Given $A, B > 0$, we have, for $x^{b_2}(\log x)^{-A} \le y \le x^{b_2}$, and $\mathfrak{S}(b_1, b_2, k)$ defined as in* (2),

$$\sum_{k\le y}\left|\sum_{n_1,n_2\le x}\Lambda(n_1^{b_1}+n_2^{b_2}+k)-\mathfrak{S}(b_1,b_2,k)x^2\right|^2 = O\left(\frac{yx^4}{(\log x)^B}\right). \quad (5)$$

As a corollary, we have that for fixed (b_1, b_2), almost all polynomials of the form $n_1^{b_1} + n_2^{b_2} + k$ with non-zero singular series capture their primes.

Corollary. *Given $A, B, C > 0$ and $\mathfrak{S}(b_1, b_2, k)$ defined as in* (2), *we have for $x^{b_2}(\log x)^{-A} \le y \le x^{b_2}$ that*

$$\sum_{n_1,n_2\le x}\Lambda(n_1^{b_1}+n_2^{b_2}+k)=\mathfrak{S}(b_1,b_2,k)x^2+O\left(\frac{x^2}{(\log x)^B}\right) \quad (6)$$

holds for all k not exceeding y with at most $O\left(y(\log x)^{-C}\right)$ exceptions.

With the above in mind, we first define the exponential sums

$$S_1(\alpha)=\sum_{m\le z}\Lambda(m)e(m\alpha), \quad (7)$$

where $\Lambda(m)$ is the von Mangoldt function and $z = x^{b_2} + x^{b_1} + y$, and

$$S_{2,l}(\alpha)=\sum_{n\le x}e(-n^l\alpha). \quad (8)$$

As in [BZ], we define the major arcs as

$$\mathfrak{M}=\bigcup_{q\le(\log x)^c}\ \bigcup_{\substack{a \bmod q\\(a,q)=1}}\left[\frac{a}{q}-\frac{1}{qQ},\frac{a}{q}+\frac{1}{qQ}\right] \quad (9)$$

where $Q = x^{1-\epsilon}$, and the minor arcs as $\mathfrak{m} = [1/Q, 1+1/Q]\backslash\mathfrak{M}$. Our starting point is the identity

$$\sum_{n_1,n_2\le x}\Lambda(n_1^{b_1}+n_2^{b_2}+k)=\int_0^1 S_1(\alpha)S_{2,b_1}(\alpha)S_{2,b_2}(\alpha)e(-k\alpha)d\alpha, \quad (10)$$

which follows from the fact that for $n \in \mathbb{Z}$,

$$\int_0^1 e(n\alpha)d\alpha=\begin{cases}1, & \text{if } n=0,\\ 0, & \text{otherwise.}\end{cases} \quad (11)$$

Using the decomposition over Dirichlet characters χ mod q

$$e\left(\frac{an}{q}\right) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(an)\tau(\bar{\chi}) \tag{12}$$

where $(an, q) = 1$ and $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{h=1}^{q} e\left(\frac{h}{q}\right)\chi(h),$$

we get that for $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}$,

$$S_1(\alpha) = T_1(\alpha) + E_1(\alpha) + O((\log z)^2) \tag{13}$$

where

$$\begin{aligned} T_1(\alpha) &= \frac{\mu(q)}{\varphi(q)} \sum_{m<z} e(\beta m) \text{ and} \\ E_1(\alpha) &= \frac{\mu(q)}{\varphi(q)} \sum_{m\le z} (\Lambda(m)-1)e(\beta m) + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \tau(\bar{\chi})\chi(a) \sum_{m\le z} \chi(m)\Lambda(m)e(\beta m). \end{aligned}$$

For what follows, note that if $\mu(q)^2 = 1$, then the condition $(n^l, q) = d$ is equivalent to $(n, q) = d$. Due to the presence of $\mu(q)$ in $T_1(\alpha)$ and $E_1(\alpha)$, we may henceforth assume that all q under consideration satisfy this. Therefore d divides n and we have $n^l/d = (n/d)^l d^{l-1}$. Thus, again using the decomposition (12), we have that for $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}$,

$$\begin{aligned} S_{2,l}(\alpha) &= \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\substack{\chi \bmod q/d \\ \chi^l=\chi_0}} \chi(-ad^{l-1})\tau(\bar{\chi}) \sum_{\substack{n\le x \\ (n^l,q)=d}} e(-\beta n^l) \\ &+ \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\substack{\chi \bmod q/d \\ \chi^l\neq\chi_0}} \chi(-ad^{l-1})\tau(\bar{\chi}) \sum_{\substack{n\le x \\ (n^l,q)=d}} \chi((n/d)^l)e(-\beta n^l) \\ &= T_{2,l}(\alpha) + E_{2,l}(\alpha), \text{ say.} \end{aligned}$$

Now

$$\sum_{\substack{\chi \bmod q/d \\ \chi^l=\chi_0}} \chi(-ad^{l-1})\tau(\bar{\chi}) = \sum_{\substack{m \bmod q/d \\ (m,q/d)=1}} e\left(\frac{-ad^{l-1}m^l}{q/d}\right). \tag{14}$$

Therefore, we have

$$T_{2,l}(\alpha) = \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\substack{m \bmod q/d \\ (m,q/d)=1}} e\left(\frac{-ad^{l-1}m^l}{q/d}\right) \sum_{\substack{n\le x \\ (n,q)=d}} e(-\beta n^l). \tag{15}$$

The main term

We first state lemmas that we will need for this section.

Lemma 1.

$$\sum_{\substack{a \bmod p\\(a,p)=1}} \sum_{r_1,r_2, \bmod p} e\left(\frac{-a(r_1^{b_1}+r_2^{b_2}+k)}{p}\right) = p(n_{p,b_1,b_2,k}-p) \tag{16}$$

where $n_{p,b_1,b_2,k}$ is defined at (2).

Lemma 2. *Let*

$$\Sigma(q) = \sum_{\substack{a \bmod q\\(a,q)=1}} \sum_{r_1,r_2, \bmod q} e\left(\frac{-a(r_1^{b_1}+r_2^{b_2}+k)}{q}\right).$$

Then we have

$$\Sigma(p_1p_2) = \Sigma(p_1)\Sigma(p_2). \tag{17}$$

Together, the above lemmas imply that for squarefree q,

$$\Sigma(q) = q\prod_{p|q}(n_{p,b_1,b_2,k}-p). \tag{18}$$

We have that

$$\begin{aligned}
&\int_{\mathfrak{M}} T_1(\alpha)T_{2,b_1}(\alpha)T_{2,b_2}(\alpha)e(-\alpha k)d\alpha \qquad (19)\\
= &\sum_{q\le(\log x)^c}\sum_{d_1,d_2|q}\sum_{\substack{a \bmod q\\(a,q)=1}} \frac{\mu(q)e(-ak/q)}{\varphi(q)\varphi(q/d_1)\varphi(q/d_2)}\\
\times &\sum_{\substack{h_1 \bmod q/d_1\\(h_1,q/d_1)=1}} e\left(\frac{-ad_1^{b_1-1}h_1^{b_1}}{q/d_1}\right)\sum_{\substack{h_2 \bmod q/d_2\\(h_2,q/d_2)=1}} e\left(\frac{-ad_2^{b_2-1}h_2^{b_2}}{q/d_2}\right)\\
\times &\int_{|\beta<1/(qQ)|}\sum_{m<z}e(\beta m)\sum_{\substack{n_1\le x\\(n_1,q)=d_1}} e(-\beta n_1^{b_1})\sum_{\substack{n_2\le x\\(n_2,q)=d_2}} e(-\beta n_2^{b_2})e(-\beta k)d\beta.
\end{aligned}$$

The integral over β is

$$\begin{aligned}
&\int_0^1\sum_{m<z}e(\beta m)\sum_{\substack{n_1\le x\\(n_1,q)=d_1}} e(-\beta n_1^{b_1})\sum_{\substack{n_2\le x\\(n_2,q)=d_2}} e(-\beta n_2^{b_2})e(-\beta k)d\beta\\
+ \ &O\left(\int_{1/qQ}^{1/2}\sum_{m<z}e(\beta m)\sum_{\substack{n_1\le x\\(n_1,q)=d_1}} e(-\beta n_1^{b_1})\sum_{\substack{n_2\le x\\(n_2,q)=d_2}} e(-\beta n_2^{b_2})e(-\beta k)d\beta\right),
\end{aligned}$$

where the O-term is, by Cauchy's inequality, bounded from above by

$$\begin{aligned} & qQ\left(\int_0^1\left|\sum_{\substack{n_1\le x\\(n_1,q)=d_1}} e(-\beta n_1^{b_1})\right|^2 d\beta\right)^{1/2}\left(\int_0^1\left|\sum_{\substack{n_2\le x\\(n_2,q)=d_2}} e(-\beta n_2^{b_2})\right|^2 d\beta\right)^{1/2} \\ \ll\ & \frac{qQx}{(d_1d_2)^{1/2}}. \qquad (20)\end{aligned}$$

Furthermore,

$$\int_0^1\sum_{m<z}e(\beta m)\sum_{\substack{n_1\le x\\(n_1,q)=d_1}} e(-\beta n_1^{b_1})\sum_{\substack{n_2\le x\\(n_2,q)=d_2}} e(-\beta n_2^{b_2})e(-\beta k)d\beta=\frac{\varphi(q/d_1)\varphi(q/d_2)x^2}{q^2}. \qquad (21)$$

We have that

$$\sum_{d_1,d_2|q}\sum_{\substack{h_1 \bmod q/d_1\\(h_1,q/d_1)=1}} e\left(\frac{-ad_1^{b_1-1}h_1^{b_1}}{q/d_1}\right)\sum_{\substack{h_2 \bmod q/d_2\\(h_2,q/d_2)=1}} e\left(\frac{-ad_2^{b_2-1}h_2^{b_2}}{q/d_2}\right)=\sum_{r_1,r_2,\bmod q} e\left(\frac{-a(r_1^{b_1}+r_2^{b_2})}{q}\right). \qquad (22)$$

Therefore, by (18), (20), (21) and (22), we get that (19) is

$$\begin{aligned} & \sum_{q\le(\log x)^c}\frac{\mu(q)\prod_{p|q}(n_{p,b_1,b_2,k}-p)}{q\varphi(q)}x^2+O\left(Qx(\log x)^{c_1}\right) \qquad (23)\\ =\ & \mathfrak{S}(b_1,b_2,k)x^2+O\left(\sum_{q>(\log x)^c}\frac{\mu(q)\prod_{p|q}(n_{p,b_1,b_2,k}-p)}{q\varphi(q)}x^2+Qx(\log x)^{c_1}\right) \qquad (24)\end{aligned}$$

for some $c_1>0$. Henceforth, we denote the tail end of the singular series by

$$\Phi(b_1,b_2,k)\ =\ \sum_{q>(\log x)^c}\frac{\mu(q)\prod_{p|q}\left(n_{p,b_1,b_2,k}-p\right)}{q\varphi(q)}. \qquad (25)$$

We see that the Hasse-Weil bound (3) readily gives an $O\left(\frac{1}{(\log x)^{c_2}}\right)$ bound, for some constant $c_2>0$, for $\Phi(b_1,b_2,k)$. Namely, we have that

$$\Phi(b_1,b_2,k)\ll\sum_{q>(\log x)^c}\frac{1}{q^{3/2-\epsilon}}\ll\frac{1}{(\log x)^{c_2}}. \qquad (26)$$

The error terms from the major arcs

We shall need the following lemmas of Gallagher and Wolke/Mikawa for this section:

Lemma 3 (Gallagher). *Let $2 < \Delta < N/2$ and $N < N' < 2N$. For arbitrary $a_n \in \mathbf{C}$, we have*

$$\int_{|\beta|<\Delta^{-1}} \left| \sum_{N<n<N'} a_n e(\beta n) \right|^2 d\beta \ll \Delta^{-2} \int_{N-\Delta/2}^{N'} \left| \sum_{\max(t,N)<n<\min(t+\Delta/2,N')} a_n \right|^2 dt$$

where the implied constant is absolute.

Proof. This is Lemma 1 in [G] in slightly modified form.
We shall also need the following lemma:

Lemma 4 (Wolke, Mikawa). *Let*

$$\mathfrak{J}(q,\Delta) = \sum_{\chi \bmod q} \int_N^{2N} \left| \sideset{}{^\sharp}\sum_{t<n<t+q\Delta} \chi(n)\Lambda(n) \right|^2 dt$$

where the $\sharp$ over the summation symbol means that if $\chi = \chi_0$, then $\chi(n)\Lambda(n)$ is replaced by $\Lambda(n) - 1$. Let ϵ, A and $B > 0$ be given. If $q \leq (\log N)^B$ and $N^{1/5+\epsilon} < \Delta < N^{1-\epsilon}$, then we have

$$\mathfrak{J}(q,\Delta) \ll (q\Delta)^2 N (\log N)^{-A} \tag{27}$$

where the implied constant depends only on ϵ, A and B.
Proof. This is Lemma 2 in [M] and can be proven using the techniques in [W3].
We now treat the terms with integrands $T_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)$, $T_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)$, $T_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)$, $E_1(\alpha)T_{2,b_1}(\alpha)T_{2,b_2}(\alpha)$, $E_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)$, $E_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)$ and $E_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)$ in similar fashion to section 6 of [BZ]. Namely, first we observe that the following bounds hold by Bessel's inequality:

$$
\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} T_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2 \ll \int_{\mathfrak{M}} \left|T_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)\right|^2 d\alpha,\\
= &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|T_1\left(\frac{a}{q}+\beta\right)T_{2,b_1}\left(\frac{a}{q}+\beta\right)E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{1/x^{b_2/2}<|\beta|<1/(qQ)}\left|T_1\left(\frac{a}{q}+\beta\right)T_{2,b_1}\left(\frac{a}{q}+\beta\right)E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
\ll &\; z^2x^2\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\; x^{b_2+2}\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/(qQ)}\left|E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} T_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2 \ll \int_{\mathfrak{M}} \left|T_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)\right|^2 d\alpha,\\
= &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|T_1\left(\frac{a}{q}+\beta\right)E_{2,b_1}\left(\frac{a}{q}+\beta\right)T_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{1/x^{b_2/2}<|\beta|<1/(qQ)}\left|T_1\left(\frac{a}{q}+\beta\right)E_{2,b_1}\left(\frac{a}{q}+\beta\right)T_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
\ll &\; z^2x^2\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|E_{2,b_1}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\; x^{b_2+2}\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/(qQ)}\left|E_{2,b_1}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta, \qquad (29)
\end{aligned}
$$

$$\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} T_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2 \ll \int_{\mathfrak{M}} \left|T_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)\right|^2 d\alpha,\\
= &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|T_1\left(\frac{a}{q}+\beta\right)E_{2,b_1}\left(\frac{a}{q}+\beta\right)E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{1/x^{b_2/2}<|\beta|<1/(qQ)}\left|T_1\left(\frac{a}{q}+\beta\right)E_{2,b_1}\left(\frac{a}{q}+\beta\right)E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
\ll &\; z^2x^2\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/x^{b_2/2}}\left|E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta\\
+ &\; x^{b_2+2}\sum_{q\le(\log x)^c}\sum_{\substack{a \bmod q\\(a,q)=1}}\int_{|\beta|<1/(qQ)}\left|E_{2,b_2}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta, \qquad (30)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} E_1(\alpha)T_{2,b_1}(\alpha)T_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2\\
\ll &\int_{\mathfrak{M}}\left|E_1(\alpha)T_{2,b_1}(\alpha)T_{2,b_2}(\alpha)\right|^2 d\alpha\\
\ll &\; x^4\int_{\mathfrak{M}}\left|E_1(\alpha)\right|^2 d\alpha, \qquad (31)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} E_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2\\
\ll &\int_{\mathfrak{M}}\left|E_1(\alpha)T_{2,b_1}(\alpha)E_{2,b_2}(\alpha)\right|^2 d\alpha\\
\ll &\; x^4\int_{\mathfrak{M}}\left|E_1(\alpha)\right|^2 d\alpha, \qquad (32)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k\le x^{b_2}}\left|\int_{\mathfrak{M}} E_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2\\
\ll &\int_{\mathfrak{M}}\left|E_1(\alpha)E_{2,b_1}(\alpha)T_{2,b_2}(\alpha)\right|^2 d\alpha\\
\ll &\; x^4\int_{\mathfrak{M}}\left|E_1(\alpha)\right|^2 d\alpha \qquad (33)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{k\le x^{b_2}} \left| \int_{\mathfrak{M}} E_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)e(-k\alpha)d\alpha \right|^2 \\
&\ll \int_{\mathfrak{M}} \left|E_1(\alpha)E_{2,b_1}(\alpha)E_{2,b_2}(\alpha)\right|^2 d\alpha \\
&\ll x^4 \int_{\mathfrak{M}} \left|E_1(\alpha)\right|^2 d\alpha. \qquad (34)
\end{aligned}
$$

We now treat all terms with integrals of the form

$$\int_{|\beta|<\Delta} \left| E_{2,b_1}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta$$

similarly to the portion of [BZ] from equation 6.2 to equation 6.4, namely, by applying the lemma of Gallagher, Lemma 3, to obtain

$$\int_{|\beta|<\Delta} \left| E_{2,b_1}\left(\frac{a}{q}+\beta\right)\right|^2 d\beta \ll \frac{x\Delta^2}{(\log x)^{c_3}} \qquad (35)$$

for some $c_3 > 0$. From this, we obtain that (28), (29) and (30) are all majorized by

$$\frac{z^2 x^{1-b_2}}{(\log x)^{c_4}} + \frac{x^{b_2+3}}{Q^2(\log x)^{c_5}} \qquad (36)$$

for some $c_4, c_5 > 0$. Furthermore, we treat all terms with integrals of the form

$$\int_{\mathfrak{M}} \left|E_1(\alpha)\right|^2 d\alpha$$

as in equation 6.7 of [BZ], namely, by applying the lemma of Wolke/Mikawa, Lemma 4, to obtain

$$
\begin{aligned}
\int_{\mathfrak{M}} \left|E_1(\alpha)\right|^2 d\alpha &\ll \sum_{q\le(\log x)^c} \frac{q}{\varphi(q)}(qQ)^{-2}\mathfrak{J}(q,Q/2) + (\log x)^{3c+2}Q \\
&\ll z(\log x)^{c-A}, \qquad (37)
\end{aligned}
$$

where $\mathfrak{J}(q,\Delta)$ is from Lemma 4, for any $A > 0$. From this we obtain that (31), (32), (33) and (34) are all majorized by

$$\frac{zx^4}{(\log x)^{c_6}} \qquad (38)$$

for some $c_6 > 0$.

The minor arcs

For this section, we need a lemma for bounding $S_{2,l}(\alpha)$ on the minor arcs.

Lemma 5.

$$|S_{2,l}(\alpha)|^{2^{l-1}} \leq (2x)^{2^{l-1}} x^{-l} \sum_{-x<r_1,r_2,\ldots,r_{l-1}<x} \min\left(x, \frac{1}{\|\alpha l! \prod_{i=1}^{l-1} r_i\|}\right). \quad (39)$$

Proof. This is Proposition 8.2 in [IK].

Using the lemma above, we proceed as follows to treat $S_{2,l}(\alpha)$ with $\alpha \in \mathfrak{m}$: First, by Dirichlet approximation, there exists a rational approximation to α of type

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{2l!x^{l-1}q}$$

with $1 \leq q \leq 2l!x^{l-1}$. Since $\alpha \in \mathfrak{m}$, we can assume $q > (\log x)^c$. Now for $-x < r_1, r_2, \ldots, r_{l-1} < x$,

$$\left|l! \prod_{i=1}^{l-1} r_i \alpha - l! \prod_{i=1}^{l-1} r_i \frac{a}{q}\right| \leq \frac{1}{2q} \Rightarrow \frac{1}{\|\alpha l! \prod_{i=1}^{l-1} r_i\|} \leq \frac{2}{\|(a/q) l! \prod_{i=1}^{l-1} r_i\|}.$$

Then we have

$$\sum_{-x<r_1,r_2,\ldots,r_{l-1}<x} \min\left(x, \frac{1}{\|\alpha l! \prod_{i=1}^{l-1} r_i\|}\right)$$
$$\ll \sum_{\substack{-x<r_1,r_2,\ldots,r_{l-1}<x \\ q | l! \prod_{i=1}^{l-1} r_i}} x + \sum_{\substack{-x<r_1,r_2,\ldots,r_{l-1}<x \\ q \nmid l! \prod_{i=1}^{l-1} r_i}} \frac{2}{\|(a/q) l! \prod_{i=1}^{l-1} r_i\|} \quad (40)$$

The number of integers of the form $\prod_{i=1}^{l-1} r_i$ with $|r_i| \leq x$ which are divsible by q is majorized by $\tau_{l-1}(q)\frac{x^{l-1}}{q}$, where $\tau_n(q)$ is the number of ways to write q as the product of n integers. We have that $\tau_n(q) \ll \tau(q)^n$, and since $\tau(q) = O(q^\epsilon)$, so is $\tau_n(q)$. Therefore,

$$\sum_{\substack{-x<r_1,r_2,\ldots,r_{l-1}<x \\ q | l! \prod_{i=1}^{l-1} r_i}} x \ll \frac{x^l}{q^{1-\epsilon}} \ll \frac{x^l}{(\log x)^{c(1-\epsilon)}} \quad (41)$$

since $q > (\log x)^c$. We also have that

$$\sum_{\substack{-x<r_1,r_2,\ldots,r_{l-1}<x \\ q\nmid l!\prod_{i=1}^{l-1} r_i}} \frac{2}{\|(a/q)l!\prod_{i=1}^{l-1} r_i\|} \ll \sum_{\substack{r=1 \\ q\nmid r}}^{x^{l-1}} \tau_{l-1}(r)\frac{1}{\|(r/q)\|}$$
$$\ll \tau_{l-1}(x^{l-1})\sum_{\substack{r=1 \\ q\nmid r}}^{x^{l-1}} \frac{1}{\|(r/q)\|}$$
$$\ll x^{l-1+\epsilon}. \tag{42}$$

Hence, we have that

$$\sup_{\alpha\in\mathfrak{m}} |S_{2,l}(\alpha)| \ll \frac{x}{(\log x)^{c_7}} \tag{43}$$

for some $c_7 > 0$. Finally, by Bessel's inequality and Cauchy's inequality, we have that

$$\sum_{k\le x^{b_2}} \left|\int_{\mathfrak{m}} S_1(\alpha)S_{2,b_1}(\alpha)S_{2,b_2}(\alpha)e(-k\alpha)d\alpha\right|^2 \ll \int_{\mathfrak{m}} |S_1(\alpha)S_{2,b_1}(\alpha)S_{2,b_2}(\alpha)|^2\,d\alpha$$
$$\ll \sup_{\alpha\in\mathfrak{m}}|S_{2,b_1}|^2 \sup_{\alpha\in\mathfrak{m}}|S_{2,b_2}|^2 \int_0^1 |S_1(\alpha)|^2\,d\alpha$$
$$\ll \frac{x^2}{(\log x)^{c_8}}\frac{x^2}{(\log x)^{c_9}} z\log z$$
$$\ll \frac{zx^4}{(\log x)^{c_{10}}} \tag{44}$$

for some constants $c_8, c_9, c_{10} > 0$. Now, combining (23), (26), (36), (38) and (44), we obtain the theorem.

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